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VIBRATION OF A STAMP PARTIALLY ADHERING TO AN ELASTIC MEDIUM

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There is examined the problem of vibration of a stamp of arbitrary planform occupying a space Ω and vibrating harmonically in an elastic medium with plane boundaries. It is assumed that the elastic medium is a packet of layers with parallel boundaries, at rest in the stiff or elastic half-space. Contact of three kinds is realized under the stamp: rigid adhesion in the domain Ω_1, friction-free contact in domain Ω_2, there are no tangential contact stresses, and "film" contact without normal force in domain Ω_3 (there are no normal contact stresses, only tangential stresses are present.). It is assumed that the boundaries of all the domains have twice continuously differentiable curvature and $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.

The problem under consideration assumes the presence of a static load pressing the stamp to the layer and hindering the formation of a separation zone. Moreover, a dynamic load, harmonic in time, acts on the stamp causing dynamical stresses which are of the greatest interest since the solution of the static problem is obtained as a particular case of the dynamic problem for $\omega = 0$ (ω is the frequency of vibration). The general solution is constructed in the form of a sum of static and dynamic solutions.

A uniqueness theorem is established for the integral equation of the problem mentioned and for the case of axisymmetric vibration of a circular stamp partially coupled rigidly to the layer, partially making friction-free contact, the problem is reduced to an effectively solvable system of integral equations of the second kind, which reduce easily to a Fredholm system.

These results are an extension of the method elucidated in [1], where by the approach in [1] must be altered qualitatively to obtain them.

1. The system of integral equtions of the problem described above has the form

$$\sum_{n=1}^{3} \iint\limits_{\Omega_1 \cup \Omega_{3-[n/3]}} k_{mn}(x-\xi, y-\eta)\, q_n(\xi, \eta)\, d\xi d\eta = f_m(x, y) \tag{1.1}$$

$$m = 1, 2, 3$$

$$k_{mn}(\xi, \eta) = \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} K_{mn}(\alpha, \beta)\, e^{-i(\alpha\xi+\beta\eta)} d\alpha\, d\beta, \quad (x, y) \in \Omega_1 \cup \Omega_{3-[m/3]}$$

Here q_1, q_2 are the tangential contact stresses with a carrier in $\Omega_1 \cup \Omega_3$ projected on Ox and Oy, respectively, q_3 is the normal contact stress with a carrier in $\Omega_1 \cup \Omega_2$, f_1 and f_2 are amplitude values of the displacement of points under the stamp in the direction of the Ox and Oy axes, respectively, given in $\Omega_1 \cup \Omega_3$, and f_3 is the analogous normal displacement given in $\Omega_1 \cup \Omega_2$.

Elements of the matrix $\mathbf{K}(\alpha, \beta)$ are given by the relationships

$$K_{11}(\alpha, \beta) = \alpha^2 M(u) + \beta^2 N(u), \quad K_{22}(\alpha, \beta) = \beta^2 M(u) + \alpha^2 N(u)$$
$$K_{12}(\alpha, \beta) = K_{21}(\alpha, \beta) = [M(u) - N(u)]\alpha\beta$$
$$i\alpha^{-1}K_{13}(\alpha, \beta) = -i\alpha^{-1}K_{31}(\alpha, \beta) = i\beta^{-1}K_{23}(\alpha, \beta) =$$
$$-i\beta^{-1}K_{32}(\alpha, \beta) = P(u)$$
$$K_{33}(\alpha, \beta) = R(u), \quad u = \sqrt{\alpha^2 + \beta^2}$$

The functions M, N, P, R are even in u. Their form is determined by the kind of medium with which the stamp makes contact. In particular, if the medium is an elastic layer coupled rigidly to an undeformable base, the following relationships hold:

$$M(u) = {}^1/_2\varkappa_2^2 u^{-2}(\sigma_2 \operatorname{sh} 2\sigma_2 \operatorname{ch} 2\sigma_1 - \sigma_1^{-1}u^2 \operatorname{sh} 2\sigma_1 \operatorname{ch} 2\sigma_2)\Delta^{-1}(u)$$
$$N(u) = 2u^{-2}\sigma_2^{-1} \operatorname{th} 2\sigma_2$$
$$P(u) = \{(2u^2 - {}^1/_2\varkappa_2^2)(1 - \operatorname{ch} 2\sigma_1 \operatorname{ch} 2\sigma_2) + \sigma_1^{-1}\sigma_2^{-1} \times$$
$$[2u^4 - (\varkappa_1^2 + {}^3/_2\varkappa_2^2) + \varkappa_1^2\varkappa_2^2]\}\Delta^{-1}$$
$$R(u) = {}^1/_2\varkappa_2^2(\sigma_1 \operatorname{sh} 2\sigma_1 \operatorname{ch} 2\sigma_2 - u^2\sigma_2^{-1} \operatorname{sh} 2\sigma_2 \operatorname{ch} 2\sigma_1)\Delta^{-1}(u)$$
$$\Delta(u) = u^2(2u^2 - \varkappa_2^2) - \left(2u^4 - u^2\varkappa_2^2 + \frac{1}{4}\varkappa_2^2\right)\operatorname{ch} 2\sigma_1 \operatorname{ch} 2\sigma_2 +$$
$$\sigma_1^{-1}\sigma_2^{-1}u^2[2u^4 - u^2(2\varkappa_2^2 + \varkappa_1^2) + \varkappa_1^2\varkappa_2^2 + {}^1/_4\varkappa_2^4] \operatorname{sh} 2\sigma_1 \operatorname{sh} 2\sigma_2$$
$$\varkappa_1^2 = \omega^2\rho h^2(\lambda + 2\mu)^{-1}, \quad \varkappa_2^2 = \omega^2\rho h^2\mu^{-1}, \quad \sigma_k = (u^2 - \varkappa_k^2)^{1/2}$$

Here λ, μ are the elastic moduli, ρ is the material density, ω is the frequency of stamp vibration, and h is half the layer thickness.

The contours Γ_1, Γ_2 in (1.1) are located in conformity with the rules set down in [2].

A uniqueness theorem analogous to that established in [1] is valid for the system of integral equations (1.1).

Let $\pm\zeta_r$ $(r = 1, 2, \ldots, p)$ denote the poles of the functions $M(u)$, $P(u)$, $R(u)$, and $\pm\eta_s$ $(s = 1, 2, \ldots, q)$ the poles of the function $N(u)$.

T h e o r e m 1. Let the domain Ω be convex. Then the system of integral equations (1.1) cannot have more than one solution in L_α, $\alpha > 1$, if $M(u)$, $N(u)$, $P(u)$, $R(u)$ possess the properties

1°. $[M^{-1}(\zeta_r)]' > 0$, $[N^{-1}(\eta_s)]' > 0$, $r = 1, 2, \ldots, p$; $s = 1, 2, \ldots, q$

2°. $[R^{-1}(\zeta_r)]'[M^{-1}(\zeta_r)]' - \{[P^{-1}(\zeta_r)]'\}^2 > 0$, $r = 1, 2, \ldots, p$

3°. There exists a matrix $\Pi(u)$ with elements $\Pi_{mn}(u)$ which are rational functions bounded at infinity, with poles at the points $\pm\zeta_r$, and $\pm\eta_s$ such that for any u $(-\infty < u < \infty)$ the real Hermitian component of the matrix $\mathbf{K}(u)\Pi^{-1}(u)$ is positive-definite.

The rather cumbersome proof of this theorem is omitted since it duplicates the method of proving Theorem 1 in [1] to a significant extent. The method of proving analogous theorems is elucidated in [1, 2].

2. Let us examine a particular case of the system (1.1), namely: Let us consider that Ω is a circle of radius a_2, the domain Ω_2 is a circle of radius a_1, and the domain Ω_1 is a ring with the inner and outer radii a_1 and a_2, respectively.

In this case the system of integral equations is simplified and takes the form

$$\int_{a_1}^{a_2} r_{k1}(r,\rho)\,q_1(\rho)\,\rho d\rho + \int_0^{a_2} r_{k2}(r,\rho)\,q_2(\rho)\,\rho d\rho = f_k(r), \quad k = 1,2 \tag{2.1}$$

$$r_{mn}(r,\rho) = \int_\gamma K_{mn}(u)\,J_{2-m}(ur)\,J_{2-n}(u\rho)\,udu$$

$$r \in [a_1, a_2], \quad k = 1; \ r \in [0, a_2], \ k = 2$$

In contrast to the approach in [1], the method of left-sided regularization based on using the analytic properties of the Fourier transform is not applied successfully in this case. Hence, in this paper an extension is given of the method of right-sided regularization of a system of integral equations, which is used in [3] in its simplest variation.

In conformity with this approach, let us set up the general form of the solution of a system of integral equations. To this end, we continue the right side of the system of integral equations outside the domain of definition. Let $\varphi(r)$ denote the continuation of the function $f_1(r)$ in the domain $r < a_1$, and $\psi_1(r)$ and $\psi_2(r)$ the continuations of the functions $f_1(r)$ and $f_2(r)$ in the domain $r > a_2$, respectively.

Let us henceforth assume that the following factorizations are performed

$$\mathbf{K} = \mathbf{K}_-\mathbf{K}_+, \quad K_{22} = K_{22}^+K_{22}^-$$

Now applying the appropriate Bessel transformations to the integral equations on the axis, we find the general representation of the solution of the system of integral equations in the following form:

$$\mathbf{q}(r) = \int_\gamma \mathbf{J}(ur)\,\mathbf{K}^{-1}(u)\,\mathbf{F}(u)\,udu + \frac{1}{2}\int_\Gamma \mathbf{H}(ur)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Z}_1(u)\,udu + \tag{2.2}$$

$$\int_\Gamma \mathbf{J}(ur)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Z}_2(u)\,udu, \quad r \in [a_1, a_2]$$

$$q_2(r) = \int_\gamma \frac{J_0(ur)\,F_2(u)}{K_{22}(u)}\,udu + \int_\Gamma \frac{J_0(ur)\,Z_5(u)}{K_{22}^+(u)}\,udu, \quad r \in [0, a_1]$$

Here $\mathbf{J}(ur)$ and $\mathbf{H}(ur)$ are diagonal matrices with the diagonal elements $J_1(ur)$, $J_0(ur)$ and $H_1^{(2)}(ur)$, $H_0^{(2)}(ur)$, respectively. The contour γ is the part of the contour Γ lying in the right half-plane. In addition, we used the notation

$$\mathbf{F}(u) = \left\{\int_{a_1}^{a_2} f_1(\rho)\,J_1(u\rho)\,\rho d\rho, \quad \int_0^{a_2} f_2(\rho)\,J_0(u\rho)\,\rho d\rho\right\} = \{F_1, F_2\}$$

$$\mathbf{Z}_1(u) = \{Z_1(u),\ Z_2(u)\},\quad \mathbf{Z}_2(u) = \{Z_3(u),\ Z_4(u)\}$$
$$\mathbf{q}(r) = \{q_1(r),\ q_2(r)\}$$

The functions $Z_k(u)$ in this latter representation require definition. To define them we introduce $q_k(r)$ into the left side of the system of integral equations and integrate. We first use a representation of the form

$$r_{mn}(r,\rho) = \int_\Gamma K_{mn}(t)\begin{Bmatrix} H_{2-m}^{(2)}(tr)\,J_{2-n}(t\rho), & r>\rho \\ H_{2-n}^{(2)}(t\rho)\,J_{2-m}(tr), & r<\rho\end{Bmatrix} tdt \tag{2.3}$$

for each kernel of the system of integral equations.

Simple manipulations reduce the relationships obtained to define the unknown $Z_k(u)$ to the following system of integral equations of the first kind:

$$\begin{aligned}&\int_\Gamma \mathbf{\Theta}_1(u,t,a_2)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Z}_2(u)\,udu = \\ &\quad -\int_\gamma \mathbf{\Theta}_1(u,t,a_2)\,\mathbf{K}^{-1}(u)\,\mathbf{F}(u)\,udu - \frac{1}{2}\int_\Gamma \mathbf{\Theta}_2(u,t,a_2)\times \\ &\quad \mathbf{K}_+^{-1}(u)\,\mathbf{Z}_1(u)\,udu \\ &\quad \frac{1}{2}\int_\Gamma \mathbf{\Theta}_3(u,t,a_1)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Z}_1(u)\,udu = -\int_\gamma \mathbf{\Theta}_4(u,t,a_1)\times \\ &\quad \mathbf{K}^{-1}(u)\,\mathbf{F}(u)\,udu - \int_\Gamma \mathbf{\Theta}_4(u,t,a_1)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Z}_2(u)\,udu \\ &\int_\Gamma \frac{Z_5(u)\,\Theta_5(u,t,a_1)}{K_{22}^+(u)}\,udu = -\int_\gamma \frac{F_2(u)\,\Theta_5(u,t,a_1)}{K_{22}(u)}\,udu\end{aligned} \tag{2.4}$$

Here $\mathbf{\Theta}_k(u,t,a)$ are diagonal second-order matrices whose first diagonal elements are, respectively, the functions

$$\frac{uH_1^{(2)}(ta)\,J_0(ua) - tH_0^{(2)}(ta)\,J_1(ua)}{t^2-u^2},\quad \frac{uH_1^{(2)}(ta)\,H_0^{(2)} - tH_0^{(2)}(ta)\,H_1^{2}(ua)}{t^2-u^2} \tag{2.5}$$
$$\frac{uJ_1(ta)\,H_0^{(2)}(ua) - tJ_0(ta)\,H_1^{(2)}(ua)}{t^2-u^2},\quad \frac{uJ_1(ta)\,J_0(ua) - tJ_0(ta)\,J_1(ua)}{t^2-u^2}$$

The second diagonal elements can be obtained from the first by the mutual replacement of the factors u and t in the first and second terms of the numerator in (2.5). Moreover we used the notation

$$\Theta_5(u,t,a_1) = \frac{tH_1^{(2)}(ta_1)\,J_0(ua_1) - uJ_1(ua_1)\,H_0^{(2)}(ta_1)}{t^2-u^2}$$

Let us study the behavior of the kernels of the integral equations obtained for large values of the parameters t and u. A simple analysis based on the use of asymptotic formulas for the Bessel functions will permit clarifying the following asymptotic relationships

$$\Theta_1(u,t,a_2)\sim\frac{i\exp[-ia_2(t-u)]}{\pi a_2\sqrt{tu}}\frac{\mathbf{E}}{t-u}$$
$$\Theta_2(u,t,a_2)\sim\frac{2\exp[-ia_2(t+u)]}{\pi a_2\sqrt{tu}}\frac{\mathbf{I}}{t+u}$$
$$\Theta_3(u,t,a_1)\sim\frac{-i\exp ia_1(t-u)}{\pi a_1\sqrt{tu}}\frac{\mathbf{E}}{t-u}$$
$$\Theta_4(u,t,a_1)\sim\frac{\exp ia_1(t+u)}{2\pi a_1\sqrt{tu}}\frac{\mathbf{I}}{t+u}$$
$$\Theta_5(u,t,a_1)\sim\frac{i\exp[-ia_1(t-u)]}{\pi a_1\sqrt{tu}}\frac{1}{t-u}$$
$$|u|\to\infty,\quad |t|\to\infty,\quad \operatorname{Im}u<0,\quad \operatorname{Im}t<0$$

Here $\mathbf{E}$ is the unit matrix, $\mathbf{I}$ is the diagonal matrix with elements 1 and -1.

Let us introduce the normalizing functions $\varkappa_1(x,a)$ and $\varkappa_2(x,a)$ which possess the asymptotic behavior

$$i\varkappa_1(x,a)\sim b\pi\sqrt{ax}\,e^{iax},\quad \varkappa_2(x,a')\sim b^{-1}\sqrt{ax}\,e^{-iax};\quad a,b=\text{const}$$

for $|x|\to\infty$, $\operatorname{Im}x<0$.

In particular, the following can be taken as these normalizing functions:

$$\varkappa_1(x,a)=\frac{\sqrt{a}}{iH_0^{(2)}(ax)},\quad \varkappa_2(x,a)=\pi x\sqrt{a}\,H_0^{(2)}(ax)$$

Let us multiply the first equation of the system (2.4) by $\mathbf{K}_+(t)\varkappa_1(t,a_2)$ the second by $\mathbf{K}_+(t)\varkappa_2(t,a_1)$, and the third by $K_{22}^+(t)\varkappa_1(t,a_1)$ by first introducing the new unkowns by means of the formulas

$$\mathbf{Z}_1^*(u)=\varkappa_1^{-1}(u,a_1)u\mathbf{Z}_1(u),\quad \mathbf{Z}_2^*(u)=\varkappa_2^{-1}(u,a_2)u\mathbf{Z}_2(u)$$
$$Z_5^*(u)=\varkappa_2^{-1}(u,a_1)uZ_5(u)$$

Let us afterwards project the relationships obtained onto the domain above the contour Γ.

The kernels of the integral equations degenerate into bisingular integrals for large values of the parameters u,t. Let us use the properties of these integrals which are associated with the possibility of their being inverted, namely: let us append the limiting bisingular integrals obtained to the left side and subtract. As a result of the manipulations mentioned and the use of the properties of the bisingular integrals, we arrive at a system of integral equations of the second kind, the first of which is

$$\mathbf{X}^+(z)+\frac{1}{4\pi^2}\int_{\Gamma_+}\int_{\Gamma}\mathbf{K}_+(t)\mathbf{M}_1(t,u,a_2)\mathbf{K}_+^{-1}(u)\mathbf{X}^+(u)\frac{du\,dt}{(t-z)(t^2-u^2)}= \tag{2.6}$$
$$-\int_{\Gamma_+}\int_{\gamma}\mathbf{K}_+(t)\Theta_1(u,t,a_2)\mathbf{K}^{-1}(u)\varkappa_1(t,a_2)\mathbf{F}(u)u\frac{du\,dt}{t-z}-$$

$$\frac{1}{8\pi^2}\int_{\Gamma_+}\int_{\Gamma}\mathbf{K}_+(t)\,\mathbf{\Theta}_2(u,t,a_2)\,\mathbf{K}_+^{-1}(u)\,\mathbf{Y}^+(u)\,\varkappa_1(t,a_2)\,\varkappa_1(u,a_1)\frac{du\,dt}{t-z}$$

The remaining equations in the unkowns $\mathbf{Y}^+(z)$ and $Z^+(z)$ are written analogously to (2.6).

The contour Γ_+ is located above the contour Γ, and z lies above Γ_+. The regular functions $\mathbf{X}^+(z)$, $\mathbf{Y}^+(z)$, $Z^+(z)$ in the domain above Γ are connected to the unkowns $\mathbf{Z}_1^*(z)$, $\mathbf{Z}_2^*(z)$, $Z_5^*(z)$ by the following relationships:

$$\mathbf{Z}_1^*(z) = \frac{1}{4\pi^2}\mathbf{Y}^+(z) + \frac{1}{2\pi i}\mathbf{K}_+(z)\,\mathbf{R}_1^-(z)$$

$$\mathbf{Z}_2^*(z) = \frac{1}{4\pi^2}\mathbf{X}^+(z) + \frac{1}{2\pi i}\mathbf{K}_+(z)\,\mathbf{R}_2^-(z)$$

$$Z_5^*(z) = \frac{1}{4\pi^2}Z^+(z) + \frac{1}{2\pi i}K_{22}^+(z)\,R_3^-(z)$$

where $^-(z)$, $\mathbf{R}_2^-(z)$, $R_3^-(z)$ are regular in the lower half-plane. The function $\mathbf{M}_1(t, u, a_2)$ in the kernel of the integral equation (2.6) has the form

$$\mathbf{M}_1(t, u, a_2) = \mathbf{\Theta}_1(u, t, a_2)\,\varkappa_1(t, a_2)\,\varkappa_2(u, a_2)\,(t^2 - u^2) - (t + u)\,\mathbf{E}$$

The integral operators of the system (2.6) are not completely continuous in the Banach space $c\,(1)$; here $c\,(\lambda)$ is the space of functions which are continuous with the weight z^λ on the contour Γ. However, manipulations can be made analogous to those used in [3], which reduce the operators mentioned to Fredholm operators. For the practical purposes of solving the mentioned system of equations there is no need to perform the mentioned manipulations since the form written down is most convenient for the solution.

In order to solve the system of integral equations (2.6) approximately, the contours Γ_+, Γ in the representations of the kernels must be deformed in the lower half-plane. Consequently, the integral over the deformed contours turns out to be small, and can be neglected, and the system of integral equations reduces to a system of algebraic equations [4].

We obtain the following relationships to determine the contact stresses:

$$\mathbf{q}(r) = \int_{\gamma}\mathbf{J}(ur)\,\mathbf{K}^{-1}(u)\,\mathbf{F}(u)\,u\,du + \frac{1}{8\pi^2}\int_{\Gamma}\varkappa_1(u,a_1)\,\mathbf{H}(ur)\times \tag{2.7}$$

$$\mathbf{K}_+^{-1}(u)\,\mathbf{Y}^+(u)\,du + \frac{1}{8\pi^2}\int_{\Gamma}\varkappa_2(u,a_2)\,\mathbf{J}(ur)\,\mathbf{K}_+^{-1}(u)\,\mathbf{X}^+(u)\,du$$

$$a_1 \leqslant r \leqslant a_2$$

$$q_2(r) = \int_{\gamma}\frac{F_2(u)\,J_0(ur)\,u\,du}{K_{22}(u)} + \frac{1}{4\pi^2}\int_{\Gamma}\frac{\varkappa_2(u,a_1)\,J_0(ur)\,Z^+(u)}{K_{22}^+(u)}\,du$$

$$0 \leqslant r \leqslant a_1$$

It is seen that finding the solution of the system of integral equations of the second kind (2.6) completely solves the problem of seeking the contact stresses.

3. Let us study the properties of the solutions of the system of integral equations.

T h e o r e m 2. The system of integral equations of the second kind (2.6) is solvable uniquely in the space of functions $c\,(1)$ if the system of integral equations (1.1) has a unique solution in L_α, $\alpha > 1$.

The proof of the theorem is based on the equivalence of the system of integral equations (1.1) to the system of integral equations with a completely continuous operator obtained from the system of integral equations of the second kind (2.6). The uniqueness of the solution of the Fredholm system of integral equations assures its solvability.

On the basis of this theorem, the general form of the solutions of the system of the second kind (2.6) is set up, namely: the asymptotic formulas

$$\mathbf{X}^+(z) \sim \mathbf{c}_1 z^{-1}, \quad \mathbf{Y}^+(z) \sim \mathbf{c}_2 z^{-1}, \quad Z^+(z) \sim c_3 z^{-1}, \quad |z| \to \infty$$

are valid.

Here $\mathbf{c}_1$, $\mathbf{c}_2$ are constant vectors, and c_3 is a constant.

Using these properties, and the relationship (2.7), we establish the following properties of the functions $q_k(r)$ (ν is the Poisson's ratio of the layer material)

$$q_1(r)(r - a_1)^{1/2}(a_2 - r)^{1/2 + i\varepsilon} \in C(a_1, a_2)$$

$$q_2(r)\,|r - a_1|^{1/2}(a_2 - r)^{1/2 + i\varepsilon} \in C(0, a_2)$$

$$\varepsilon = \frac{1}{\pi} \operatorname{arth} \frac{1 - 2\nu}{2(1 - \nu)}$$

For an approximate evaluation of stress at inner points of the domains Ω_i, deformation of the contours in (2.7) can be used. Evaluating the residues of the integrands at the poles intersected by the deformable contours, and neglecting small integral terms, we obtain the following representation for the solution:

$$\mathbf{q}(r) \approx \mathbf{f}\Big\{\mathbf{J}(\eta r)\mathbf{K}^{-1}(\eta) - \frac{i}{4\pi}\sum_{k=1}^{n} \varkappa_1(-p_k, a_1)\,\mathbf{H}(-p_k r) \times \tag{3.1}$$

$$\operatorname*{Res}_{u=-p_k} \mathbf{K}_+^{-1}(u)\,\mathbf{Y}^+(-p_k) - \frac{i}{4\pi}\sum_{k=1}^{n} \varkappa_2(-p_k, a_2)\,\mathbf{J}(-p_k r) \times$$

$$\operatorname*{Res}_{u=-p_k} \mathbf{K}_+^{-1}(u)\,\mathbf{X}^+(-p_k)\Big\}, \quad a_1 < r < a_2$$

$$q_2(r) = f_2\left\{\frac{J_0(\eta r)}{K_{22}(\eta)} - \frac{i}{2\pi}\sum_{s=1}^{m} \varkappa_1(-z_s, a_1)\,J_0(-z_s r)\,\frac{Z^+(-z_s)}{[K_{22}^+(-z_s)]'}\right\}$$

$$0 \leqslant r < a_1$$

Here $-p_k$ ($k = 1, 2, \ldots, n$), $-z_s$ ($s = 1, 2, \ldots, m$) are, respectively, the zeroes of $\det \mathbf{K}(z)$ and $K_{22}(z)$ in the lower half-plane. Moreover,

$$f_1(r) = f_1 J_1(\eta r), \quad f_2(r) = f_2 J_0(\eta r), \quad \mathbf{f} = \{f_1, f_2\}$$

$$\eta, f_i = \text{const}$$

was taken as the right sides of (2.1) without limiting the generality in obtaining the expression (3.1).

To study the behavior of the surface outside the stamp and in the domain Ω_2 the values found for $q_k(r)$ must be inserted in the system of integral equations (2.1) and the value of the right sides outside the domain of contact and the value of the right side in the first equation in the domain Ω_2 must be evaluated. Consequently, a representation of the following kind $(\Gamma_- < \Gamma)$:

$$\psi(r) = \frac{1}{2}\int_\gamma\int_{\Gamma_-} u\mathbf{H}(ur)\mathbf{K}(u)\Big[\mathbf{D}_4(u,t)\mathbf{K}^{-1}(t)\mathbf{F}(t)t + 2\mathbf{c}\frac{F_2(t)t}{K_{22}^+(t)}\times \tag{3.2}$$

$$D_3(u,t)\Big]dudt + \frac{1}{16\pi^2}\int_\Gamma\int_{\Gamma_-} u\mathbf{H}(ur)\mathbf{K}(u)\Big[\mathbf{D}_1(u,t)\mathbf{K}_+^{-1}(t)\mathbf{Y}^+(t)\times$$

$$\varkappa_1(t,a_1) + \mathbf{D}_4(u,t)\mathbf{K}_+^{-1}(t)\mathbf{X}^+(t)\varkappa_2(t,a_2) + 2\mathbf{c}\frac{Z^+(t)}{K_{22}^+(t)}\times$$

$$\varkappa_2(t,a_1)D_3(u,t)\Big]dudt \qquad r > a_2$$

$$\varphi(r) = \frac{\mathbf{a}}{2}\int_\gamma\int_{\Gamma_-} t\mathbf{J}(tr)\mathbf{K}(t)\mathbf{D}_1(u,t)\mathbf{K}^{-1}(u)\mathbf{F}(u)udt\,du +$$

$$\frac{\mathbf{a}}{16\pi^2}\int_\Gamma\int_{\Gamma_-} t\mathbf{J}(tr)\mathbf{K}(t)[\mathbf{D}_2(u,t)\mathbf{K}_+^{-1}(u)\mathbf{Y}^+(u)\varkappa_1(u,a_1) +$$

$$\mathbf{D}_1(u,t)\mathbf{K}_+^{-1}(u)\mathbf{X}^+(u)\varkappa_2(u,a_2)]\,dtdu +$$

$$\int_\gamma \frac{F_2(u)uJ_1(ur)}{K_{22}(u)}\Big[K_{12}(u) + \frac{a_1}{2}\int_{\Gamma_-} tK_{12}(t)\Theta_5(u,t,a_1)\,dt\Big]du +$$

$$\frac{1}{4\pi^2}\int_{\Gamma_-}\frac{\varkappa_2(u,a_2)Z^+(u)J_1(ur)}{K_{22}^+(u)}\times$$

$$\Big[K_{12}(u) + \frac{a_1}{2}\int_{\Gamma_-} tK_{12}(t)\Theta_5(u,t,a_1)\,dt\Big]du \qquad 0 \leqslant r < a_1$$

is obtained for the behavior of the suface.

The following notation was used here

$$\mathbf{a} = \{1, 0\},\ \mathbf{c} = \{0, 1\},\ \boldsymbol{\psi}(r) = \{\psi_1(r), \psi_2(r)\}$$

$$\mathbf{D}_k(u,t) = a_2\boldsymbol{\Theta}_k(u,t,a_2) - a_1\boldsymbol{\Theta}_k(u,t,a_1),\ k = 1, 2, 4$$

$$D_3(u,t) = \frac{a_1}{t^2-u^2}[tJ_1(ta_1)J_0(ua_1) - uJ_0(ta_1)J_1(ua_1)]$$

Let us present a theorem permitting estimation of the error in the approximate solution (3.1), (3.2) of the problem (2.1). For the sake of brevity, its proof performed by using the method of perturbations is omitted.

T h e o r e m 3. Let the system (2.1) with the matrix kernel $\mathbf{K}(u)$ have the solution $\mathbf{q} = \{q_1, q_2\}$ and with the matrix $\mathbf{K}^*(u)$ the solution $\mathbf{q}^* = \{q_1^*, q_2^*\}$. Then if the elements $K_{ij}(u)$ and $K_{ij}^*(u)$ of the matrices $\mathbf{K}(u)$ and $\mathbf{K}^*(u)$ in the conditions of Theorem 1 satisfy the conditions

$$|K_{ij}(u) - K_{ij}^*(u)|\,|K_{ij}(u)|^{-1}(1 + |u|)^\alpha < \varepsilon, \quad \alpha > 1/2$$

then for sufficiently small ε the inequality

$$\left|[q_k(r) - q_k^*(r)]\sqrt{|r - a_1|\,|r - a_2|}\right| < t\varepsilon$$

is valid where t is independent of q_k and ε.

The problems examined may be used as a model for the investigation of the nature of the contact between foundations and thè ground, as well as in the defectoscopy of glue compounds. Namely, by having the set of solutions of the problem for different contact conditions it is possible to predict the domains of incomplete coupling from the condition of best agreement between the experimental and theoretical results.

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